ADIABATIC MEASUREMENTS ON METASTABLE SYSTEMS

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In several situations, most notably when describing metastable states, a system can evolve according to an effective non hermitian Hamiltonian. To each eigenvalue of a non hermitian Hamiltonian is associated an eigenstate $|\phi\rangle$ which evolves forward in time and an eigenstate $\langle\psi|$ which evolves backward in time. Quantum measurements on such systems are analyzed in detail with particular emphasis on adiabatic measurements in which the measuring device is coupled weakly to the system. It is shown that in this case the outcome of the measurement of an observable A is the weak value $\langle\psi|A|\phi\rangle/\langle\psi|\phi\rangle$ associated to the two-state vector $\langle\psi|\ |\phi\rangle$ corresponding to one of the eigenvalues of the non hermitian Hamiltonian. The possibility of performing such measurements in a laboratory is discussed.

Any interaction between two systems can be regarded, in a very wide sense, as a "measurement" since the state of one of the systems, the measuring device, is affected by the state of the other one, the measured system. In general, however, this interaction is not very "clean", that is, the information about the properties of the measured system cannot be read easily from the final state of the measuring device. Only some very particular classes of interactions are clean enough and are called "measurements" in the usual, more restricted, sense.

The best known type of quantum measurement is the von Neumann ideal measurement wherein the system is coupled impulsively to the measuring device. The Hamiltonian describing such a measurement is

$$H = H_0 + g(t)PA + H_{MD}, \tag{1}$$

where H_0 is the free Hamiltonian of the system, H_{MD} is the free Hamiltonian of the measuring device, P is the momentum conjugate to the position variable Q of the pointer of the measuring device, A is the observable to be measured. The coupling parameter $g(t)$ is normalized to $\int g(t)dt = 1$ and is taken to be non vanishing during a very small interval Δt. Thus, the interaction term dominates the rest of the Hamiltonian during Δt, and the time evolution e^{-iPA} leads to a correlated state: eigenstates of A with eigenvalues a_n are correlated to measuring device states in which the pointer is shifted by these values a_n (here and below we use units such that $\hbar = 1$). Thus in an ideal measurements the final state of the measuring device is very simple related to the state of the measured system. The properties of ideal measurements are:

a) The outcome of the measurement can only be one of the eigenvalues a_i.

b) A particular outcome a_i appears at random, with probability depending only on the initial state of the measured system and is independent of the details of the measurement.

c) The measurement leads to the (true or effective, depending on one's preferred interpretation) collapse of the wave-function of the measured system on the eigenstate $|a_i\rangle$. Subsequent ideal measurements of the same observable A invariably yield the same eigenvalue a_i.

The opposite limit of extremely weak and long interaction is also clean enough to be called a measurement [1,2]. In such an adiabatic (or protective) measurement, the coupling is very small: $g(t) = 1/T$ for most of the interaction time T and $g(t)$ goes to zero gradually before and after the period T. In order that the measurement be as clean as possible, we also impose that: the initial state of the measuring device is such that the momentum P is bounded; that the momentum P is a constant of motion of the whole Hamiltonian eq. (1) (but we shall only consider the simpler case where H_{MD} vanishes); and that the free Hamiltonian H_0 has non-degenerate eigenvalues E_i. For $g(t)$ smooth enough we then obtain an adiabatic process in which the system cannot make a transition from one energy eigenstate to another, and, in the limit $T \to \infty$, the interaction Hamiltonian changes the energy eigenstate by an infinitesimal amount. If the initial state of the system is an eigenstate $|E_i\rangle$ of H_0 then for any given value of P, the energy of the eigenstate shifts by an infinitesimal amount given by the first order perturbation theory: $\delta E = \langle E_i|H_{int}|E_i\rangle = \langle E_i|A|E_i\rangle P/T$. The corresponding time evolution $e^{-iP\langle E_i|A|E_i\rangle}$ shifts the pointer by the expectation value of A in the state $|E_i\rangle$. The main properties of adiabatic measurements are:

a) The outcome of the measurement can only be the expectation value $\langle A\rangle_i = \langle E_i|A|E_i\rangle$.

b) A particular outcome $\langle A\rangle_i$ appears at random, with a probability which depends only on the initial state of the measured system and is independent of the details of the measurement.

c) The measurement leads to the collapse of the wave-function of the measured system on the energy eigenstate $|E_i\rangle$ corresponding to the observed expectation value $\langle A\rangle_i$ [3]. Subsequent adiabatic measurements of the same

observable A invariably yield the expectation value in the same eigenstate $|E_i\rangle$.

d) Simultaneous measurements of different observables yield the expectation value in the same energy eigenstate $|E_i\rangle$.

The aim of the present letter is to consider measurements on systems which evolve according to an effective non hermitian Hamiltonian. While ideal (impulsive) measurements on such systems lead to no surprise (since in an impulsive measurement the unperturbed Hamiltonian of the measured system plays no role), adiabatic measurements yield as outcomes some new type of values associated with the measured observable, namely the "weak values" [4]. Weak values where originally introduced in the context of the two state formalism [5,4,6,7] wherein a system is described by two states, the usual one $|\Psi_1\rangle$ evolving towards the future from the initial time t_1, and a second state $\langle\Psi_2|$ evolving towards the past from the final time t_2. If at an intermediate time a sufficiently weak measurement is carried out on such a pre- and post-selected system, the state of the measuring device after the post-selection is shifted to $\Psi_{MD}(Q) \rightarrow \Psi_{MD}(Q - A_w)$, where A_w is the weak value of the observable A

$$A_w = \frac{\langle\Psi_2|A|\Psi_1\rangle}{\langle\Psi_2|\Psi_1\rangle}. \tag{2}$$

Note that weak values can take values which lie outside the range of eigenvalues of A and are in general complex. Their real and imaginary part affect the position and momentum of the pointer respectively. Weak values are associated with two states which in the present context are the left and right eigenstates of the effective Hamiltonian (see below) [8]. The main properties of adiabatic measurements carried out on a system evolving according to an effective non hermitian Hamiltonian are:

a) The only possible outcomes of the measurement are the weak values A_w^i corresponding to one of the pairs of states $\langle\psi_i||\phi_i\rangle$ associated with the non hermitian Hamiltonian.

b) A particular outcome A_w^i appears at random, with a probability which depends only on the initial state of the measured system and is independent of the details of the measurement.

c) The measurement leads to an effective collapse to the two-state vector $\langle\psi_i||\phi_i\rangle$ corresponding to the observed weak value A_w^i. Subsequent adiabatic measurements of the same observable A invariably yield the same weak value.

d) Simultaneous measurements of different observables yield the weak values corresponding to the same two-state vector $\langle\psi_i||\phi_i\rangle$.

Although the Hamiltonian of a quantum system is always a hermitian operator, under suitable conditions a subsystem may evolve according to an effective non hermitian Hamiltonian. A well known case is the description of metastable states [9]. If the system is initially in the metastable state $\psi(0)$, after a time t it will be in the state $\psi(t) = e^{-iH_{eff}t}\psi(0) +$ *decay products* where H_{eff} is the effective non hermitian Hamiltonian. A celebrated example where this description has proved extremely useful is the Kaon system. Another case in which a system evolves according to an effective non hermitian Hamiltonian is when it is coupled to a suitably pre- and post-selected system [8]. As an example, consider a spin 1/2 particle coupled to a pre- and post-selected system S of large spin N through the Hamiltonian

$$H_0 = \lambda \mathbf{S} \cdot \sigma. \tag{3}$$

The large spin is pre-selected at t_1 to be in the state $|S_x{=}N\rangle$ and post-selected to be at t_2 in the state $\langle S_y{=}N|$. The coupling constant λ is chosen in such a way that the interaction with our spin-1/2 particle cannot change significantly the two-state vector of the system S. Indeed, the system with the spin S can be considered as N spin 1/2 particles all pre-selected in $|\uparrow_x\rangle$ state and post-selected in $|\uparrow_y\rangle$ state. Since the strength of the coupling to each spin 1/2 particle is $\lambda \ll 1$, during the time of the measurement their states cannot change significantly. (However λN must be large so that the effective Hamiltonian is significant.) Thus, the forward evolving state $|S_x{=}N\rangle$ and the backward evolving state $\langle S_y{=}N|$ do not change significantly during the measuring process. Hence, effectively, the spin-1/2 particle is coupled to the weak value of S

$$\mathbf{S}_w = \frac{\langle S_y = N|(S_x, S_y, S_z)|S_x = N\rangle}{\langle S_y = N|S_x = N\rangle} = (N, N, iN), \tag{4}$$

and the effective non hermitian Hamiltonian is given by

$$H_{eff} = \lambda N(\sigma_x + \sigma_y + i\sigma_z). \tag{5}$$

The non hermiticity of H_{eff} is due to the complexity of $\mathbf{S}_w$. A detailed discussion of this example is given below.

Note that the effective non hermitian Hamiltonians only arise due to a partial post-selection. In the spin example it only applies if the large spin is found in the state $\langle S_y = N|$. In the case of metastable states it only applies to the metastable states so long as they have not decayed.

We now analyze the general properties of a non hermitian Hamiltonian H_{eff} which has non degenerate eigenvalues ω_i. In general the eigenvalues are complex. Denote the eigenkets and the eigenbras of H_{eff} by $|\phi_i\rangle$ and $\langle\psi_i|$:

$$H_{eff}|\phi_i\rangle = \omega_i|\phi_i\rangle, \qquad \langle\psi_i|H_{eff} = \omega_i\langle\psi_i|. \tag{6}$$

Contrary to the case where H_{eff} is hermitian, the $|\phi_i\rangle$ are not orthogonal to each other, nor are the $\langle\psi_i|$, and furthermore $|\psi_i\rangle \neq |\phi_i\rangle$. However the $|\phi_i\rangle$ and $\langle\psi_i|$ each form a complete set, and they obey the mutual orthogonality condition

$$\langle\psi_i|\phi_j\rangle = \langle\psi_i|\phi_i\rangle\delta_{ij}, \qquad (7)$$

which follows from subtracting the two identities $\langle\psi_i|H_{eff}|\phi_j\rangle = \omega_j\langle\psi_i|\phi_j\rangle$, $\langle\psi_i|H_{eff}|\phi_j\rangle = \omega_i\langle\psi_i|\phi_j\rangle$ for $i \neq j$. Eq. (7) enables us to rewrite H_{eff} as

$$H_{eff} = \sum_i \omega_i \frac{|\phi_i\rangle\langle\psi_i|}{\langle\psi_i|\phi_i\rangle}, \qquad (8)$$

which generalizes the diagonalization of hermitian operators. The eigenkets of H_{eff} are the natural basis in which to decompose a forward evolving state $|\Phi\rangle$. Indeed, using the decomposition of unity $I = \sum_i \frac{|\phi_i\rangle\langle\psi_i|}{\langle\psi_i|\phi_i\rangle}$ one obtains

$$|\Phi\rangle = \sum_i \frac{\langle\psi_i|\Phi\rangle}{\langle\psi_i|\phi_i\rangle}|\phi_i\rangle = \sum_i \alpha_i|\phi_i\rangle \qquad (9)$$

(On the other hand a backward evolving state should be decomposed into the eigenbras of H_{eff} as $\langle\Psi| = \sum_i \beta_i\langle\psi_i|$). The formal solution of the Schrödinger's equation with the effective Hamiltonian H_{eff} is:

$$|\Phi(t)\rangle = e^{-iH_{eff}t}|\Phi\rangle = \sum_i \alpha_i e^{-i\omega_i t}|\phi_i\rangle \qquad (10)$$

Note that the norm $\mathcal{N}$ of $|\Phi(t)\rangle$ is not equal to 1 but is time dependent. Formally, there are two causes for not conserving the norm in time evolution due to the effective Hamiltonian. The first is that the eigenvalues ω_i may be complex. The second is that the eigenkets are not necessarily orthogonal. This non conservation of probability by non hermitian Hamiltonians has a natural interpretation when one recalls that we are describing partially post-selected systems. In the case of metastable states $\mathcal{N}(t)$ is the probability for the states not to have decayed. In the spin example $\mathcal{N}(t)$ describes corrections to the probability of finding the state $\langle S_y = N|$.

Let us illustrate this general formalism by considering the Kaon system. The two eigenkets of the effective Hamiltonian are traditionally denoted $|K_L\rangle$ and $|K_S\rangle$. Similarly, one can define the eigenbras of the effective Kaon Hamiltonian $\langle K'_L|$ and $\langle K'_S|$. The particular features of non hermitian Hamiltonians are controlled by the CP violation parameter $\epsilon \simeq 10^{-3}$. The non orthogonality of the eigenkets is $\langle K_S|K_L\rangle = O(\epsilon)$ and the non equality of the right and left eigenstates is $\langle K'_L|K_L\rangle = 1 - O(\epsilon^2)$. In view of the smallness of ϵ the adiabatic measurements which we propose below may be difficult to implement in the Kaon system. However, other metastable systems may display much stronger non orthogonality and be more amenable to experiment.

In the spin example, the effective Hamiltonian eq. (5) has two eigenvalues $+\lambda N$ and $-\lambda N$ with eigenkets (bras) $|\uparrow_x\rangle$ ($\langle\uparrow_y|$) and $|\downarrow_y\rangle$ ($\langle\downarrow_x|$) respectively. Thus, H_{eff} can be rewritten as

$$H_{eff} = \lambda N \frac{|\uparrow_x\rangle\langle\uparrow_y|}{\langle\uparrow_y|\uparrow_x\rangle} - \lambda N \frac{|\downarrow_y\rangle\langle\downarrow_x|}{\langle\downarrow_x|\downarrow_y\rangle}. \qquad (11)$$

In this example the eigenkets and eigenbras associated with the same eigenvalue are very different. Thus, weak values associated with these two states can have surprising values. For example, $\langle\downarrow_x|\sigma_z|\downarrow_y\rangle/\langle\downarrow_x|\downarrow_y\rangle = -i$, which is pure imaginary and $\langle\downarrow_x|(\sigma_x+\sigma_y)/\sqrt{2}|\downarrow_y\rangle/\langle\downarrow_x|\downarrow_y\rangle = -\sqrt{2}$, which lies outside the range of eigenvalues of $\sigma\cdot\mathbf{n}$.

We are now ready to discuss adiabatic measurements performed on a system evolving according to H_{eff}. The Hamiltonian describing such a measurement is given by eq. (1) with H_0 replaced by H_{eff}. The coupling parameter $g(t)$ equals $1/T$ for most of the interaction time T and goes to zero gradually before and after the period T. In order that the measurement be as clean as possible we also impose that: H_{eff} has non degenerate eigenvalues; that the initial state of the measuring device is such that the momentum P is bounded; and that the momentum P is a constant of motion of the whole Hamiltonian eq. (1). For $g(t)$ smooth enough, and in the limit $T\to\infty$, we obtain once more an adiabatic process such that if the system is initially in an eigenket $|\phi_i\rangle$, it will still be in the same eigenket after the measurement. Furthermore, in this limit, the interaction Hamiltonian changes the eigenket during the interaction by an infinitesimal amount.

If we take the initial state of the system to be an eigenket $|\phi_i\rangle$, then for any given value of P, the eigenvalue of the eigenstate shifts by an infinitesimal amount which can be obtained using first order perturbation theory as follows. The perturbed eigenstates are solutions of

$$\begin{aligned}&\left(H_{eff} + \tfrac{P}{T}A\right)\left(|\phi_i\rangle + \textstyle\sum_{j\neq i} c_{ij}|\phi_j\rangle\right) = \\ &\quad(\omega_i + \delta\omega_i)\left(|\phi_i\rangle + \textstyle\sum_{j\neq i} c_{ij}|\phi_j\rangle\right). \end{aligned}\qquad (12)$$

Taking the scalar product with $\langle\psi_i|$, to first order in P/T one obtains

$$\delta\omega_i = \frac{P}{T}\frac{\langle\psi_i|A|\phi_i\rangle}{\langle\psi_i|\phi_i\rangle} = \frac{P}{T}A^i_w. \qquad (13)$$

Thus the state of the measuring device after the measurement is shifted, $\Psi_{MD}(Q) \to \Psi_{MD}(Q - A^i_w)$, and if the initial wave function of Ψ_{MD} is sufficiently peaked in Q, the reading of the measuring device yields the weak value of A.

It is instructive to consider the case when the initial state is not an eigenket of H_{eff}. The initial state should then be decomposed into a superposition of eigenkets $|\Phi\rangle = \Sigma_i\alpha_i|\phi_i\rangle$ and its time evolution, up to normalization, will be given by

$$|\Phi\rangle\psi_{MD}(Q) \to \Sigma_i\alpha_i e^{-i\omega_i T}|\phi_i\rangle\psi_{MD}(Q - A^i_w). \qquad (14)$$

The state of the measuring device is amplified to a macroscopically distinguishable situation. Then, effectively, a collapse takes place to the reading of one of

the weak values A_w^i with the relative probabilities given by $|\alpha_i|^2 e^{2Im(\omega_i)T}$. We call the collapse effective because it only occurs under the condition that a partial post-selection is realized. A subsequent adiabatic measurement of another observable B will yield the weak value corresponding to the same two-state $\langle\psi_i||\phi_i\rangle$. Alternatively, one can carry out the measurements of A and B simultaneously. This can always be done by increasing the duration T of the measurement so that the interaction $(P_1A+P_2B)/T$ remains a small perturbation. Thus, given a sufficiently long time T, one can obtain reliable measurements of any set of observables by making measuring devices interact adiabatically with a single quantum system. However it should be noted that in any realistic implementation we will need ensembles of systems and measuring devices since both in the case of metastable states and in the spin example the probability of a successful partial post-selection (which gives rise to the effective non hermitian Hamiltonian) is very small. Indeed, the adiabatic measurement will only be successful if the metastable states do not decay during the measurement, or if the spin S is found in the state $|S_y = N\rangle$. Nevertheless, there is a non-zero probability that the first run with a single system and a single set of measuring devices will yield the desired outcomes.

Our general discussion was carried out for a system evolving according to an arbitrary effective non hermitian Hamiltonian. The spin example presented above is amenable to exact treatment and one can investigate in this case in what limit the effective non hermitian Hamiltonian describes adequately the evolution of the spin 1/2 particle. We recall that the effective Hamiltonian eq(5) has two eigenkets $|\uparrow_x\rangle$ and $|\downarrow_y\rangle$. That $|\uparrow_x\rangle$ should be an eigenket is easily be seen by noting that the initial state $|S_x = N\rangle|\uparrow_x\rangle$ is an eigenstate of the free Hamiltonian $H_0 = \lambda\mathbf{S}\cdot\sigma$. That $|\downarrow_y\rangle$ is an eigenket is a nontrivial prediction which can be checked by calculating the probability for the small spin, initially in the state $|\downarrow_y\rangle$, to be in the state $|\uparrow_y\rangle$ at an intermediate time. One finds that this probability is proportional to $1/N^2$, thereby confirming that it is indeed an eigenket in the limit of large N.

If the initial state of the small spin is $|\downarrow_y\rangle$, and an adiabatic measurement of $\sigma_\xi = \sigma\cdot\hat{\xi}$ is carried out the eigenket $|\downarrow_y\rangle$ should be unaffected by the measurement, and the pointer of the measuring device should be shifted by $(\sigma_\xi)_w = \frac{\langle\downarrow_x|\sigma_\xi|\downarrow_y\rangle}{\langle\downarrow_x|\downarrow_y\rangle}$. In order to verify this we considered the particular case when $\hat{\xi} = \hat{x}$ whereupon the analysis simplifies considerably since only the states with $J_x = S_x + \sigma_x = N + 1/2,\ N - 1/2,\ N - 3/2$ come up in the calculation. Thus, we took the Hamiltonian to be $H = \lambda\mathbf{S}\cdot\sigma + \frac{P}{T}\sigma_x$ during the interval $t_1 < t < t_2 = t_1 + T$, with the initial state $|S_x = N\rangle|\downarrow_y\rangle$ and the final state of the large spin post-selected to be $\langle S_y = N|$. Taking the measuring device to be in the momentum eigenstate P, one finds that after the post-selection, at $t = t_2$, the state of the small spin plus measuring device is $|\downarrow_y\rangle e^{iP/2} +$ *error terms*. The error terms are either of the form $|\downarrow_y\rangle e^{-iP/2}$ corresponding to a pointer shifted in the wrong direction, or of the form $f(P)|\uparrow_y\rangle$ corresponding to the spin not having remained in the state $|\downarrow_y\rangle$. The norm of the error terms is proportional to $1/N$ and in the limit of large N they can be neglected. One then finds that after and during the measurement the spin is still in the eigenket $|\downarrow_y\rangle$ and that the pointer of the measuring device is shifted by the weak value $(\sigma_x)_w = \frac{\langle\downarrow_x|\sigma_x|\downarrow_y\rangle}{\langle\downarrow_x|\downarrow_y\rangle} = -1$. Thus we confirm that in the limit of large N, the evolution is given by the effective non hermitian Hamiltonian.

In this letter we have analyzed adiabatic measurements on systems which evolve according to an effective non hermitian Hamiltonian. The effective Hamiltonian only arises when a partial post-selection is realized. For an adiabatic measurement to yield a significantly unusual result, the non hermiticity of the Hamiltonian must be large, and in such cases the probability of a successful partial post-selection is very small. There is however a reasonable hope of performing such a measurement in a real laboratory. It is conceivable to build an experiment in which the measuring device is a particular degree of freedom of the measured particle itself, and in this case the post-selection process is particularly simple [10].

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